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A Simplified Introduction to Music Algebra: from the Scale Vectors to the Modal Tensor

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***Abstract*—*In this paper we take a step forward towards the attainment of a formalism that allows to establish a deeper connection between Music and Algebra. Starting from the writing of the Ionian Scale as a Vector, we define the Ionian Modal Tensor. We prove that all the Scales that derive from the Ionian Mode, as well as all the corresponding Seventh Chords, herein considered as being Scalars, can be obtained from the above-mentioned Tensor by resorting to the concepts of Standard Basis and Dot Product. Moreover, by opportunely summing the Vectors of the Standard Basis to each other, we define some interesting Fundamental Vectors such as the "Monk – Powell" Vector and the "Guide Notes" one.*

Keywords*—*Music Algebra, Scale Vector, Modal Tensor, Harmonization, Standard Basis, Matrix, Dot Product.

I. BRIEF INTRODUCTION

We start by defining the *Ionian scale* a *vector* whose *components* are nothing but the *degrees* of the *scale*. The same procedure is applied to all the *scales* that derive from the above-mentioned. Subsequently, we define the *Ionian Modal Tensor* and, by resorting to the concepts of *standard basis* and *dot product*, we deduce the *seventh chords vector*. A reasonable mastery of the fundamental notions concerning *vectors* and *matrices* is required.

II. SCALES AS VECTORS

If we denote with X a generic note belonging to the *Chromatic scale*, and with t a *whole tone interval* [1] [2], we can represent the *Ionian scale* as a *vector*:

$$\boldsymbol{s}^{Ion}(X) = \left(X, X+t, X+2t, X+\frac{5}{2}t, X+\frac{7}{2}t, X+\frac{9}{2}t, X+\frac{11}{2}t\right) \quad (1)$$

For example, by setting $X = C$, from *(1)* we banally obtain:

$$\boldsymbol{s}^{Ion}(C) = (C, D, E, F, G, A, B) \quad (2)$$

Obviously, whatever scale can be represented as a vector. In this regard, let's consider all the *modes* (or *scales*) that can be deduced from the *Ionian* one (*Dorian*, *Phrygian*, *Lydian*, *Mixolydian*, *Aeolian*, *Locrian*) [3] [4]. Bearing in mind the *Ionian harmonization*, and denoting with $\boldsymbol{s}^{Ion,n}$ the *scale vector* derived from the *n-th degree* of the *Ionian scale* (n is a positive integer that runs from 1 to 7), we can write, with obvious meaning of the notation, the following:

$$\boldsymbol{s}^{Ion,1}(X) = \boldsymbol{s}^{Ion}(Y) \qquad Y = X \quad (3)$$

$$\boldsymbol{s}^{Ion,2}(X) = \boldsymbol{s}^{Dor}(Y) \qquad Y = X + t \quad (4)$$

$$\boldsymbol{s}^{Ion,3}(X) = \boldsymbol{s}^{Phr}(Y) \qquad Y = X + 2t \quad (5)$$

$$\boldsymbol{s}^{Ion,4}(X) = \boldsymbol{s}^{Lyd}(Y) \qquad Y = X + \frac{5}{2}t \quad (6)$$

$$\boldsymbol{s}^{Ion,5}(X) = \boldsymbol{s}^{Mix}(Y) \qquad Y = X + \frac{7}{2}t \quad (7)$$

$$\boldsymbol{s}^{Ion,6}(X) = \boldsymbol{s}^{Aeo}(Y) \qquad Y = X + \frac{9}{2}t \quad (8)$$

$$\boldsymbol{s}^{Ion,7}(X) = \boldsymbol{s}^{Loc}(Y) \qquad Y = X + \frac{11}{2}t \quad (9)$$

Clearly, the opposite procedure can be easily followed: in other terms, we can choose any *mode* (among those to date considered) and determine the *Ionian scale* from which it derives (if we select a *mode* and set Y, we can determine X). For example, since the *Dorian mode* arises from the *second degree* of the *Ionian scale* [3] [4] [5] [6] [7], taking into account *(4)*, we can immediately write:

$$\boldsymbol{s}^{Dor}(Y) = \boldsymbol{s}^{Ion,2}(X) \qquad X = Y - t \quad (10)$$

If we set $X = C$, from *(3)*, *(4)*, *(5)*, *(6)*, *(7)*, *(8)* and *(9)* we obtain, respectively:

$$\boldsymbol{s}^{Ion,1}(C) = \boldsymbol{s}^{Ion}(C) = (C, D, E, F, G, A, B) \quad (11)$$

$$\boldsymbol{s}^{Ion,2}(C) = \boldsymbol{s}^{Dor}(D) = (D, E, F, G, A, B, C) \quad (12)$$

$$\boldsymbol{s}^{Ion,3}(C) = \boldsymbol{s}^{Phr}(E) = (E, F, G, A, B, C, D) \quad (13)$$

$$\boldsymbol{s}^{Ion,4}(C) = \boldsymbol{s}^{Lyd}(F) = (F, G, A, B, C, D, E) \quad (14)$$

$$\boldsymbol{s}^{Ion,5}(C) = \boldsymbol{s}^{Mix}(G) = (G, A, B, C, D, E, F) \quad (15)$$

$$\boldsymbol{s}^{Ion,6}(C) = \boldsymbol{s}^{Aeo}(A) = (A, B, C, D, E, F, G) \quad (16)$$

$$\boldsymbol{s}^{Ion,7}(C) = \boldsymbol{s}^{Loc}(B) = (B, C, D, E, F, G, A) \quad (17)$$

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III. THE MODAL TENSOR

Taking into account (11), (12), (13), (14), (15), (16) and (17), we instantly obtain a matrix of notes whose rows and columns are nothing but the scale vectors that derive from the *Ionian mode*:

$$\boldsymbol{M}^{Ion}(C) = \begin{bmatrix} C & D & E & F & G & A & B \\ D & E & F & G & A & B & C \\ E & F & G & A & B & C & D \\ F & G & A & B & C & D & E \\ G & A & B & C & D & E & F \\ A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \end{bmatrix} \quad (18)$$

The above-mentioned matrix represents the *Ionian modal tensor* denoted by $\boldsymbol{M}^{Ion}$. We highlight that the "***M***" is written in bold, since we do not deal with a *scalar*, and in upper-case in order to create a distinction between tensors and vectors. The latter are herein written, in fact, in lower-case. The *Ionian modal tensor* is *symmetric*: consequently, with obvious meaning of the notation, we have:

$$M_{ij}^{Ion} = M_{ji}^{Ion} \quad (19)$$

As a consequence, if, once again, we denote with $\boldsymbol{s}^{Ion,n}$ the *scale vector* derived from the *n-th degree* of the *Ionian scale*, we can evidently write:

$$\boldsymbol{s}^{Ion,n} = \left(M_{n1}^{Ion}, M_{n2}^{Ion}, M_{n3}^{Ion}, M_{n4}^{Ion}, M_{n5}^{Ion}, M_{n6}^{Ion}, M_{n7}^{Ion}\right) \quad (20)$$

$$\boldsymbol{s}^{Ion,n} = \left(M_{1n}^{Ion}, M_{2n}^{Ion}, M_{3n}^{Ion}, M_{4n}^{Ion}, M_{5n}^{Ion}, M_{6n}^{Ion}, M_{7n}^{Ion}\right) \quad (21)$$

Let's now consider the *standard basis* of the 7-dimensional *Euclidean space* ($\mathscr{R}^7$):

$$\boldsymbol{d}^1 = (1,0,0,0,0,0,0) \quad (22)$$

$$\boldsymbol{d}^2 = (0,1,0,0,0,0,0) \quad (23)$$

$$\boldsymbol{d}^3 = (0,0,1,0,0,0,0) \quad (24)$$

$$\boldsymbol{d}^4 = (0,0,0,1,0,0,0) \quad (25)$$

$$\boldsymbol{d}^5 = (0,0,0,0,1,0,0) \quad (26)$$

$$\boldsymbol{d}^6 = (0,0,0,0,0,1,0) \quad (27)$$

$$\boldsymbol{d}^7 = (0,0,0,0,0,0,1) \quad (28)$$

The *standard basis* consists in seven *fundamental vectors*. Evidently, every *fundamental vector* is characterized by having all the *components* null except for the one specified in the superscript, which is equal to 1.

At this point, it is easy to imagine how a generic *scale vector* $\boldsymbol{s}^{Ion,n}$ can be obtained by means of the *dot product* between the *modal tensor* and the *n-th fundamental vector*:

$$\boldsymbol{s}^{Ion,n}(X) = \boldsymbol{M}^{Ion}(X) \cdot \boldsymbol{d}^n \quad (29)$$

For example, taking into account *(5)* and *(29)*, we have:

$$\boldsymbol{s}^{Phr}(Y) = \boldsymbol{s}^{Ion,3}(X) = \boldsymbol{M}^{Ion}(X) \cdot \boldsymbol{d}^3 \qquad Y = X + 2t \quad (30)$$

From *(30)*, by setting $X = C$, we immediately obtain:

$$\boldsymbol{s}^{Phr}(E) = \boldsymbol{s}^{Ion,3}(C) = \boldsymbol{M}^{Ion}(C) \cdot \boldsymbol{d}^3 \quad (31)$$

$$\boldsymbol{s}^{Phr}(E) = \begin{bmatrix} C & D & E & F & G & A & B \\ D & E & F & G & A & B & C \\ E & F & G & A & B & C & D \\ F & G & A & B & C & D & E \\ G & A & B & C & D & E & F \\ A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} E \\ F \\ G \\ A \\ B \\ C \\ D \end{bmatrix} \quad (32)$$

Obviously, the *fundamental vectors* can be summed to each other. According to our notation, the vectors we obtain are characterized by superscripts that reveal the non-null components (the components equal to 1). We have:

$$\boldsymbol{d}^1 + \boldsymbol{d}^7 = \boldsymbol{d}^{17} = (1,0,0,0,0,0,1) \quad (33)$$

$$\boldsymbol{d}^3 + \boldsymbol{d}^7 = \boldsymbol{d}^{37} = (0,0,1,0,0,0,1) \quad (34)$$

$$\boldsymbol{d}^1 + \boldsymbol{d}^3 + \boldsymbol{d}^5 = \boldsymbol{d}^{135} = (1,0,1,0,1,0,0) \quad (35)$$

$$\boldsymbol{d}^1 + \boldsymbol{d}^3 + \boldsymbol{d}^5 + \boldsymbol{d}^7 = \boldsymbol{d}^{1357} = (1,0,1,0,1,0,1) \quad (36)$$

We can now name the vectors we have just deduced. Very intuitively, $\boldsymbol{d}^{17}$ may be named *"Monk-Powell" fundamental vector* (since both *Thelonious Monk* [8] [9] and *Bud Powell* [10] [11] used to play *dyads*, exclusively consisting of the *root* and the *seventh* of the *chord*, with the left hand), $\boldsymbol{d}^{37}$ *"guide notes" fundamental vector*, $\boldsymbol{d}^{135}$ *triad fundamental vector*, $\boldsymbol{d}^{1357}$ *seventh chord fundamental vector*.

In particular, the vector $\boldsymbol{d}^{1357}$ may be exploited to obtain seventh chords (that we regard as *scalars*) from scales, by resorting, once again, to the *dot product*. If $\boldsymbol{s}^{Ion,n}(X)$ is the *vector* that represents the *mode* derived from the *n-th degree* of the *Ionian Scale* of *X*, the corresponding *seventh chord* can be immediately obtained, with obvious meaning of the notation, as follows:

$$c^{Ion,n}(X) = \boldsymbol{s}^{Ion,n}(X) \cdot \boldsymbol{d}^{1357} = \sum_{i=1}^{7} s_i^{Ion,n}(X)\, d_i^{1357} \quad (37)$$

For example, taking into account *(6)* and *(37)*, we have:

$$c^{Lyd}(Y) = \boldsymbol{c}^{Ion,4}(X) = \boldsymbol{s}^{Ion,4}(X) \cdot \boldsymbol{d}^{1357} \qquad Y = X + \frac{5}{2}t \quad (38)$$

From *(38)*, by setting $X = C$, we obtain:

$$c^{Lyd}(F) = \boldsymbol{s}^{Ion,4}(C) \cdot \boldsymbol{d}^{1357} = F + A + C + E = Fmaj7 \quad (39)$$

As it can be easily imagined, we can also consider *vectors* whose *components* consist in *chords*. On this subject, let's denote with $\boldsymbol{h}^{Ion}(X)$ the *vector* whose *components* are nothing but the *seventh chords* that arise from the *harmonization* of the *Ionian Scale* of *X*. It is very easy to verify that the above-mentioned vector can be banally obtained as follows:

$$\boldsymbol{h}^{Ion}(X) = \boldsymbol{M}^{Ion}(X) \cdot \boldsymbol{d}^{1357} \quad (40)$$

From the previous identity, by setting, for example, $X = C$, we instantly obtain:

$$\boldsymbol{h}^{Ion}(C) = \boldsymbol{M}^{Ion}(C) \cdot \boldsymbol{d}^{1357} \quad (41)$$

More explicitly, from *(41)* we can write:

$$\boldsymbol{h}^{Ion}(C) = \begin{bmatrix} C & D & E & F & G & A & B \\ D & E & F & G & A & B & C \\ E & F & G & A & B & C & D \\ F & G & A & B & C & D & E \\ G & A & B & C & D & E & F \\ A & B & C & D & E & F & G \\ B & C & D & E & F & G & A \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} Cmaj7 \\ Dm7 \\ Em7 \\ Fmaj7 \\ G7 \\ Am7 \\ B\emptyset \end{bmatrix} \quad (42)$$

IV. FINAL REMARKS

Although the approach herein briefly described can be considered, at least to a certain measure, concretely innovative, we underline, for the sake of clarity, that the representation of a scale as a vector (a *Pitch-Class Set*) is anything but a novelty. [12] [13] For example, the *Ionian Scale* can be alternatively represented as follows:

$$\boldsymbol{s}^{Ion} = \{0,2,4,5,7,9,11\} \quad (43)$$

The *elements* of the *set* (the *components* of the *vector*) represent the *distance* between the *scale degrees* (from the first to the seventh) and the *Tonic*: the first is null since the *distance* between the *Tonic* and itself is obviously equal to zero, the second is equal to 2 since the distance between the *Supertonic* and the *Tonic* is equal to 2 *semitones*, the third is equal to 4 since the distance between the *Mediant* and the *Tonic* is equal to 4 *semitones*, and so on.

Finally, it is worth highlighting that the line of reasoning herein followed can be applied to whatever heptatonic scale: by starting, for example, from the *Ipoionian Scale*, we can easily obtain the corresponding *Modal Tensor* $\boldsymbol{M}^{Ipo}$ and the *Harmonization Vector* $\boldsymbol{h}^{Ipo}$.

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